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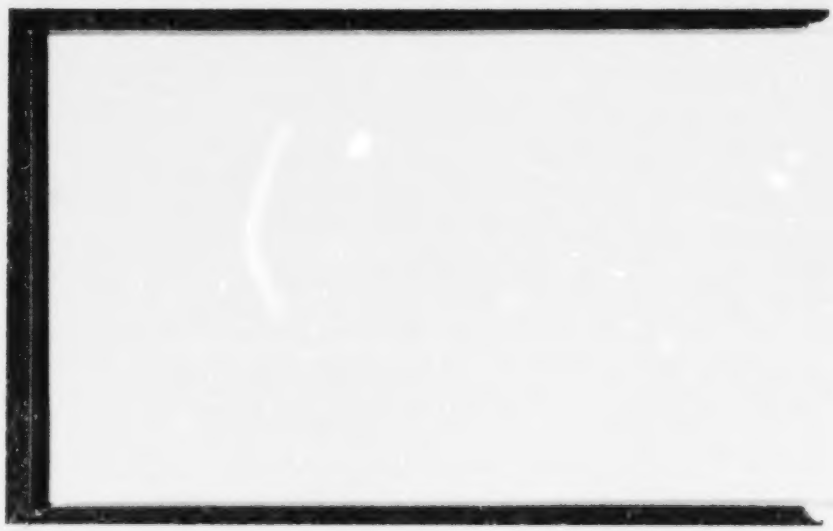

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SPATIAL ALLOCATION ON A NETWORK WITH CONGESTION

by

Marc Los and Sang Nguyen

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ABSTRACT

This article formalizes the land use design problem as a discrete-convex programming problem integrating within a quadratic assignment framework a realistic representation of transportation behavior (automobile congestion and variable demand for travel) as modelled by a combined trip distribution trip assignment model. Hill-climbing algorithms are proposed to solve the resulting optimization problem. Their performance is compared and evaluated on a set of test problems.

RÉSUMÉ

Cet article formalise la planification d'un plan d'utilisation des sols comme problème de programmation discrète-convexe intégrant dans le cadre d'un problème d'affectation quadratique une représentation réaliste du comportement en matière de transport (congestion automobile et demande variable de déplacements) qui se traduit par un modèle combiné de distribution et d'affectation des déplacements. Des algorithmes de hill-climbing sont proposés pour résoudre le problème d'optimisation résultant. Leurs performances respectives sont comparées et évaluées sur un ensemble de problèmes-tests.

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I Introduction: the problem of optimal land use design

Several approaches have been proposed so far for the modelling of a land use design problem by mathematical programming. Some are continuous approaches,[1] others are discrete.[2] Continuous approaches are best suited for the allocation of land use activities at a broad scale level while discrete approaches are best suited for detailed land use planning. This comes from the fact that economies of scale or minimum size requirements cannot be ignored at a high level of disaggregation, while they can be ignored at an aggregate level. The discrete approaches proposed have been various extensions of the quadratic assignment problem.[3]

The land use planning problem is a multiple-objective problem. One of these objectives is the minimization of total travel cost which depends simultaneously on the demand for travel and on travel costs themselves. One of the major weaknesses of previous modelling efforts at land use plan design has been the very crude modelling of transportation behavior in an urban transportation system. If we assume that there is only one mode of transport, the private automobile, transportation behavior on a road network with congestion is characterized by two aspects:

- travel costs are not constant but depend on flows on the different

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1. Coelho and Williams (1977).
2. Brotchie and Sharpe (1974), Hopkins (1977), Hopkins and Los (1979), Los (1979)
3. Koopmans and Beckmann (1957), Lawler (1963), Los (1980).

links of the road network. In *equilibrium traffic assignment* the flows and the travel costs are consistent with the so-called "user-equilibrium" principle.[1]

- The demand for travel between O-D pairs on the road network is given by a spatial interaction model of the gravity type.[2] Demands between O-D pairs consistent with user-equilibrium travel costs are the solution of the so-called *combined trip distribution trip assignment* model.[3]

The purpose of this paper is to present methods for the land use design problem which integrate transportation behavior as modelled by a combined trip-distribution trip-assignment model within a quadratic assignment formulation. In this fashion the allocation of urban activities obtained would be consistent with the transportation behavior which it generates.

We will proceed in two steps. Firstly we will introduce transportation congestion but will keep the assumption of fixed demand for travel between O-D pairs: this is the simplest departure from the quadratic assignment problem. Secondly we will consider the more realistic problem where there is also variable demand for travel and where some trips occur with congestion (during peak hours) and others without congestion (outside peak hours). The mathematical programs we thus obtain are *discrete-convex programming* problems (Geoffrion, 1975) of a very difficult nature. In order to solve them we develop heuristic techniques of the hill-climbing type

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1. See Florian (1976) for a detailed presentation and discussion of traffic equilibrium concepts and algorithms and Florian and Nguyen (1975) for an empirical validation.
2. Wilson (1970).
3. Evans (1976); Florian, Nguyen and Ferland (1975); Frank (1978).

(Reiter and Sherman, 1965) inspired by some hill-climbing techniques developed and tested for quadratic assignment problems of the Koopmans-Beckmann type (Los, 1980). In this paper we will assume that the road network is given and that there is only one mode of transport, the private car.

The rest of the paper is organized as follows: section II states the land use design problem with fixed demand for travel, the assumptions of the problem, proposes two solution algorithms and presents the results of some numerical experiments. Section III states a land use design problem with variable demand for travel given by a combined trip distribution, trip assignment model. It then proposes two solution algorithms and presents the results of the numerical tests. Section IV discusses several possible extensions and concludes the paper.

II A land-use design model with fixed demand for travel

A. Mathematical formulation and assumptions

We assume that the land-use design problem consists in assigning n discrete activities to n discrete locations, represented by specific nodes on a transportation network with congestion effects, so as to minimize a weighted sum of total site cost. and total cost of travel.

The problem is the following:

$$\text{LUFU}^{1}: \min_{\{y_{ik}\}} \phi = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}\, y_{ik} + \sum_{a \in A} v_a^*\, s_a(v_a^*) \qquad (1)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} y_{ik} = 1 \qquad \text{all } k=1..n \qquad (2)$$

---

1. For Land Use design with Fixed demand for travel and User-equilibrium.

$$\sum_{k=1}^{n} y_{ik} = 1 \qquad \text{all } i=1..n \tag{3}$$

$$y_{ik} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to location } k \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where the $\{v_a^*\}$ are the solution to the following inner minimization problem:

$$\text{U.E.A.}^1: \quad \underset{\{h_{m,k\ell}\}}{\text{Min}} \quad \sum_{a \in A} \int_0^{v_a} s_a(x)dx \tag{5}$$

$$\text{s.t.} \quad \sum_m h_{m,k\ell} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, y_{ik}\, y_{j\ell} \qquad \text{all } k=1..n, \ \ell=1..n \tag{6}$$

$$\text{where} \quad v_a = \sum_m \sum_k \sum_\ell \delta_{am,k\ell}\, h_{m,k\ell} \qquad \text{all } a \in A \tag{7}$$

$$\text{and} \quad \delta_{am,k\ell} = \begin{cases} 1 & \text{if } a \text{ belongs to path } m \text{ for O-D pair } (k,\ell) \\ 0 & \text{otherwise} \end{cases}$$

$$h_{m,k\ell} \geq 0 \tag{8}$$

The matrix $B = [b_{ik}]$ $i=1..n$, $k=1..n$ is given: b_{ik} represents the site cost of assigning activity i to location k. It can be the sum of a development cost, a spatial congestion cost (preventing building above a certain height in certain areas) and an ecological cost as long as the latter depends on the relationship between the given activity and the given site and not on the locations of other activities.

A is the set of links a of the transportation network, v_a is the number of vehicles on link a and $s_a(v_a)$ is the volume-delay function on link a assumed to be a continuously differentiable increasing function of the flow v_a on the link.

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1. U.E.A. = User-Equilibrium-Assignment.

The functions $s_a(v_a)$ are assumed known.

$Q = [q_{ij}]$ i=1..n, j=1..n is the given matrix of fixed demands for travel between activity pairs. It is measured in person-trips and if we assume a uniform car-occupancy factor, it is easily converted into vehicle-trips. In constraint (6) and in the rest of the paper, it is assumed that the uniform car-occupancy factor equals one, but this in no way restricts the land use design models proposed here.

The quantity $\sum_i \sum_k b_{ik}\, y_{ik}$ represents the total site cost for a given plan, while $\sum_{a \in A} v_a^*\, s_a(v_a^*)$ represents the total cost of travel, under the assumption of a user-equilibrium traffic assignment. Constraints (2) and (3) state that there is one and only one activity per location. The trips between k and ℓ, i.e. $\sum_i \sum_j q_{ij}\, y_{ik}\, y_{j\ell}$ subdivide into path flows $h_{m,k\ell}$ on each possible path between locations k and ℓ and constraint (6) is a conservation of flows constraint.

The mathematical program to solve is a mixed non-linear programming problem whose structure has the following properties. For a given land-use plan, i.e. for given y_{ik}, i=1..n, k=1,..n, the problem reduces to a user-equilibrium assignment problem. If the volume-delay functions were constant, the user-equilibrium problem U.E.A. would reduce to an all-or-nothing traffic assignment problem, total travel cost could then be expressed as $\sum_i \sum_j \sum_k \sum_\ell q_{ij}\, c_{k\ell}\, y_{ik}\, y_{j\ell}$ where $c_{k\ell}$ would represent the constant travel cost on a shortest path between k and ℓ. The land use design problem would reduce to a quadratic assignment problem of the Koopmans-Beckmann type.

The model makes the following assumptions:

1) There is only one mode of travel: the private automobile.
2) All trips occur during peak hours.
3) The road network is assumed given, with all the link characteristics.
4) Land use is treated in a discrete fashion and the size of each activity must be determined before applying the model: a given land use (say open space) must be divided into a given number of discrete modules (i.e. activities).
5) The demand for travel between location pairs is assumed fixed and independent of the travel costs between location pairs. This is a strong assumption which is far from reality. It will be relaxed in the next section of the paper.

If we assumed that the flows were given by a system-optimal pattern instead of a user-equilibrium, the land-use design problem LUFU would be replaced by the following simpler problem:

$$\text{LUFO}^{1}: \quad \underset{\{y_{ik}\} \text{ and } \{v_a\}}{\text{Min}} \quad \phi = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}\, y_{ik} + \sum_{a \in A} v_a\, s_a(v_a) \tag{9}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} y_{ik} = 1 \qquad k=1..n \tag{10}$$

$$\sum_{k=1}^{n} y_{ik} = 1 \qquad i=1..n \tag{11}$$

$$\sum_{m} h_{m,k\ell} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, y_{ik}\, y_{j\ell} \qquad \text{all}(k,\ell) \tag{12}$$

---

1. LUFO, for Land Use design with Fixed demand and system-Optimal flows.

$$v_a = \sum_m \sum_k \sum_\ell \delta_{am,k\ell}\, h_{m,k\ell} \qquad \text{all } a \tag{13}$$

$$\delta_{am,k\ell} = \begin{cases} 1 & \text{if a belongs to path m for O-D pair } (k\ell) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

$$y_{ik} = \begin{cases} 1 \text{ if activity i is assigned to location k} \\ 0 \text{ otherwise} \end{cases}$$

$$h_{m,k\ell} \geq 0 \tag{15}$$

B. Solution algorithms

Two heuristic techniques are proposed to solve the land use design problem (1) - (8). The first method (called ALLCØS) approximates the problem by attempting to solve the "system-optimal" approximation (9) - (15). The second one (called ALLCØU) attempts to solve directly problem (1) - (8).

1. A heuristic technique for LUFO: ALLCØS

This hill-climbing algorithm produces a local optimum which depends on the initial solution. It works as follows:

a) Initialization: generate a random spatial allocation of the n activities, compute the system-optimal flows and the value of the objective function for the initial solution.

b) For each one of the n(n-1)/2 pairwise exchanges of activities, compute a lower bound on the value of the objective function if the exchange were implemented. This lower bound is used as an estimate of that value.

c) Sort the possible exchanges in the order of non-decreasing values of the estimates. If all estimates are at least as great as the value of the best current solution, stop. Otherwise test for implementation the

exchange with the least estimate: i.e. apply a traffic assignment algorithm which gives a system optimal flow pattern. If there is indeed a decrease in the objective function, implement that exchange, update the best current solution and go to step b). If not, test for implementation the next exchange in the list etc... When a prespecified number of exchanges, ME (which is a parameter of the computer program), has been tested without success the algorithm stops.

The estimate is $\phi_1 + \Delta F + \Delta S$ where ϕ_1 is the cost of the current solution, ΔF is the change in the sum of site costs (which can be computed exactly) and ΔS is an estimate of the change in total travel cost given by:

$$\Delta S = \sum_{k=1}^{n} \sum_{\ell=1}^{n} u^*_{k\ell} \, \Delta q_{k\ell} \qquad (16)$$

where the $u^*_{k\ell}$ are the current values of the dual variables of the network assignment problem and $\Delta q_{k\ell}$ is the change in the number of trips between locations k and ℓ, were the exchange of activities currently examined to be implemented. In appendix 2 it is proved that the estimate is indeed a lower bound.

The computation of a system-optimal flow pattern requires the application of a convex programming technique. The code used here is due to Nguyen [1] and is an implementation of the Frank and Wolfe method. The main rationale for computing an estimate of the cost after exchange, rather than an exact value, is to avoid having to apply a traffic assignment technique for each exchange tested. The estimate has two roles: firstly

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1. Sang Nguyen: "A Unified Approach to Equilibrium Methods for Traffic Assignment" in Florian (1976).

to guide the search towards the best exchanges and secondly to eliminate exchanges which cannot lead to improvements. The first role assumes that the lower bounds are ordered in the same order as the true costs. This is only approximately so; hence the necessity of testing at least a pre-specified number of "promising" exchanges, i.e. exchanges with a small lower bound. The second role is based on the very fact that the estimate is a lower bound.

2. A heuristic technique for LUFU: ALLCØU

This hill-climbing algorithm produces a local optimum. It works by iterating between a pairwise interchange procedure for the Koopmans-Beckmann problem and a user-equilibrium traffic assignment technique. It proceeds as follows:

a) Initialization: generate a random spatial allocation of the n activities and compute the user-equilibrium flows and the value of the objective function of the initial solution.

b) Assuming that the travel costs between origin-destination pairs remain constant and equal to their current values, generate a local optimum to the resulting Koopmans-Beckmann problem, by applying a pairwise interchange technique.

c) Compute the flows and the value of the objective function for the spatial allocation obtained in step b). If there is an improvement of the objective function, update the current best spatial allocation and go to step b. If no improvement of the objective function occurs, stop.

A code developed by Los (1980) (called ACTIV) was used in the implementation

of step b, while a code developed by Nguyen implementing the Frank and Wolfe method was used to compute user-equilibrium flows.

C. Numerical experiments

The algorithms ALLCØS and ALLCØU were both implemented on the CDC Cyber 173 of the Université de Montréal. The results are reported in tables 1, 2 and 3. ALLCØS is clearly an expensive method, all the more so when the parameter ME increases. ALLCØU on the other hand is quite efficient. The solutions obtained are comparable in quality. It can be noticed from table 1 that for this example the assumption of a system-optimal traffic assignment is not too restrictive since the values of the solutions obtained are close when computed with the user-equilibrium assumption (3rd column) or the system-optimal pattern assumption (2nd column). It can also be checked that a value of ME between 3 and 6, i.e. between 1/12 and 1/6 of the total number of potential exchanges ($\frac{n(n-1)}{n}$) seems enough in order to obtain good local optima.

To summarize, ALLCØU is preferable to ALLCØS for solving problem LUFU if we take into account the efficiency and solution quality.

Value of the initial solution	Value of the local optimum	Cost of the locally optimal permutation computed with a user-equilibrium assignment	Execution time in seconds of CDC Cyber 173	Numbers of exchanges implemented
525 019	209 397	215 352	131.9	4
300 425	206 622	208 704	131.6	4
460 305	216 934	217 355	106.2	3
307 961	207 936	211 087	184.2	6
297 336	222 571	227 871	106.4	3
277 546	236 298	245 625	54.5	1

Table 1

Results obtained with ALLCØS on the data set # 1
with six different initial permutations
generated at random, with ME = 9 [1]

1. TRAFFIC, the code for system-optimal traffic assignment, was used with at most 20 iterations and a "tolerance level" ε of .02

ME	1	2	3	6	9	12
Value of the initial allocation	292 602	292 602	292 602	292 602	292 602	292 602
Value of the local optimum	175 178	171 273	171 273	171 273	171 273	171 273
Execution time in sec. of CDC Cyber 173	20.3	48.8	51.8	89.1	122.1	152.3
Value of the initial allocation	217 451	217 451	217 451	217 451	217 451	217 451
Value of the local optimum	178 712	171 273	171 273	171 273	171 273	171 273
Execution time in sec. of CDC Cyber 173	14.4	43.0	60.1	91.7	110.6	120.0
Value of the initial allocation	272 683	272 683	272 683	272 683	272 683	272 683
Value of the local optimum	178 404	178 712	181 784	178 712	178 712	178 712
Execution time in sec. of CDC Cyber 173	17.2	37.3	54.6	129.2	171.9	212.3
Value of the initial allocation	231 440	231 440	231 440	231 440	231 440	231 440
Value of the local optimum	214 084	177 847	177 847	177 847	177 847	177 847
Execution time in sec. of CDC Cyber 173	8.7	31.8	46.0	83.5	114.9	137.8
Value of the initial allocation	227 375	227 375	227 375	227 375	227 375	227 375
Value of the local optimum	195 066	195 066	189 417	186 531	186 531	186 531
Execution time in sec. of CDC Cyber 173	11.5	20.0	37.2	106.2	140.6	158.0
Value of the initial allocation	204 089	204 089	204 089	204 089	204 089	204 089
Value of the local optimum	183 139	183 139	174 728	174 728	174 728	174 728
Execution time in sec. of CDC Cyber 173	14.5	25.8	54.7	89.0	106.4	122.9

Table 2: Results obtained by ALLCØS for different values of the parameter ME and for six different initial permutations, on the data set #2.

Value of the initial solution computed as $\sum_k \sum_\ell \sum_{ij} u_{k\ell} q_{ij} y_{ik} y_{j\ell}$	Value of the local optimum computed as $\sum_k \sum_\ell \sum_{ij} u_{k\ell} q_{ij} y_{ik} y_{j\ell}$	Value of the local optimum computed as $\sum_a v_a s_a(v_a)$	Execution time in seconds of CDC Cyber 173	Number of iterations
269 828	169 002	186 536	8.6	1
216 572	161 902	176 897	14.3	3
255 386	174 478	189 415	8.7	1
219 882	172 686	186 221	8.7	1
223 276	181 578	193 131	8.7	1
201 144	172 620	186 743	11.6	2

Table 3

Results obtained with ALLCØU on the data set # 2

with six different initial permutations generated

at random (the same as in table 2)[1]

1. TRAFFIC was used with at most 20 iterations and a "tolerance level" ε of .005. This explains the difference between values in the second and third columns.

III A land-use design model with variable demand for travel

A. Mathematical formulation and assumptions

The model presented in this section is more realistic than the model of Section II in that it assumes variable demand for travel and distinguishes between peak-hour and non peak-hour trips: the demand for travel between O-D pairs during peak-hours is assumed given by a combined distribution-assignment model[1] and the demand for travel between O-D pairs during non peak-hours is given by a doubly-constrained gravity model. The total number of trips attracted and generated by each activity is exogenously given as well as the proportion which occurs during peak-hours. Outside peak-hours is is assumed that the free-flow travel times occur on the links of the road network. We keep the assumptions that there is one mode of travel, the private automobile, that land-use is treated discretely and that we know the road network as well as the link characteristics. Only a subset of the nodes of the network constitute potential sites for activities and therefore can generate or attract traffic.

The land-use design problem with variable demand for travel consists in assigning n discrete activities to n discrete locations so as to minimize a weighted sum of total site cost and total cost of travel (peak-hour and non peak-hour travel). The problem is the following:

$$\text{LUV}^2: \quad \min_{\{y_{ik}\}} \phi = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}\, y_{ik} + \sum_{p=1}^{n} \sum_{q=1}^{n} u^*_{pq}\, g_{pq}^{c^*} + \sum_{p=1}^{n} \sum_{q=1}^{n} c_{pq}\, g_{pq}^{\bar{c}^*} \qquad (17)$$

---

1. See Evans (1976) and Florian, Nguyen and Ferland (1975) for a presentation and solution algorithms for the combined distribution-assignment problem.
2. LUV means Land Use design with Variable demand.

$$\text{s.t.} \quad \sum_{l=1}^{n} y_{ik} = 1 \qquad \text{all } k=1..n \qquad (18)$$

$$\sum_{k=1}^{n} y_{ik} = 1 \qquad \text{all } i=1..n \qquad (19)$$

$$y_{ik} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to location } k \\ 0 & \text{otherwise} \end{cases}$$

In the expression of objective function (17) g_{pq}^{c*} is the demand between locations p and q which is generated by solving the combined distribution assignment problem stated below and u_{pq}^{*} represents the dual variable (or travel cost) associated with the O-D pair(p,q), i.e. the dual variable associated with constraint (21):

$$\underline{\text{CDA}}^{1}: \quad \underset{v_a, g_{pq}^{c}}{\text{Min}} \quad \sum_{a \in A} \int_{0}^{v_a} s_a(x)dx + \theta^{c} \sum_{p=1}^{n} \sum_{q=1}^{n} g_{pq}^{c} \ln g_{pq}^{c} \qquad (20)$$

$$\text{s.t.} \quad \sum_{m} h_{m,pq} = g_{pq}^{c} \qquad \text{all } (p,q) \qquad (21)$$

$$\sum_{q=1}^{n} g_{pq}^{c} = \sum_{i=1}^{n} O_i^{c} y_{ip} \qquad \text{all } p=1..n \qquad (22)$$

$$\sum_{p=1}^{n} g_{pq}^{c} = \sum_{j=1}^{n} D_j^{c} y_{jq} \qquad \text{all } q=1..n \qquad (23)$$

$$\text{where} \quad v_a = \sum_{m} \delta_{am,pq} h_{m,pq} \qquad \text{all } a \in A \qquad (24)$$

$$\delta_{am,pq} = \begin{cases} 1 & \text{if } a \text{ belongs to path } m \text{ between } p \text{ and } q \\ 0 & \text{otherwise} \end{cases}$$

$$h_{m,pq} \geq 0 \qquad \text{all } m,pq \qquad (25)$$

---

1. For Combined Distribution Assignment.

In the expression of objective function (17) $g_{pq}^{\bar{c}^*}$ is the demand between locations p and q which is given by the following doubly-constrained gravity model:

$$\text{DCG}^{1}: \quad \underset{g_{pq}^{\bar{c}}}{\text{Min}} \quad \sum_{p=1}^{n} \sum_{q=1}^{n} c_{pq}\, g_{pq}^{\bar{c}} + \theta^{\bar{c}} \sum_{p=1}^{n} \sum_{q=1}^{n} g_{pq}^{\bar{c}} \ln g_{pq}^{\bar{c}} \tag{26}$$

$$\text{s.t.} \quad \sum_{q=1}^{n} g_{pq}^{\bar{c}} = \sum_{i=1}^{n} O_i^{\bar{c}}\, y_{ip} \qquad \text{all } p=1..n \tag{27}$$

$$\sum_{p=1}^{n} g_{pq}^{\bar{c}} = \sum_{j=1}^{n} D_j^{\bar{c}}\, y_{jq} \qquad \text{all } q=1..n \tag{28}$$

$$g_{pq}^{\bar{c}} \geq 0 \qquad \text{all } (p,q) \tag{29}$$

The minimization problem (20) - (25) generates the demand for travel during peak-hours. This demand is expressed by a doubly-constrained gravity model which is *consistent* with travel costs obeying Wardrop's user-equilibrium conditions.[2] Constraint (21) expresses the conservation of flows. Constraint (22) is a trip-origin constraint which states that the total number of trips to all destinations generated at location p during peak-hours is equal to the total number of trips generated during peak-hours by the activity i which is located at location p. Constraint (23) is an analogous trip-attraction constraint for location q.

The minimization problem (26) - (29) generates the demand for travel during non peak-hours. This demand is expressed as a doubly-constrained gravity model consistent with constant travel costs c_{pq} between origin-destination

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1. For Doubly-Constrained Gravity model.
2. See Evans (1976) or Florian, Nguyen and Ferland (1975) for a proof of this statement. This proof relies simply on writing the Kuhn-Tucker conditions of CDA.

pairs. These constant travel costs are given by an all-or-nothing traffic assignment problem in which the link times are the free-flow travel times. Constraints (27) and (28) are respectively trip generation and trip attraction constraints for non peak-hour trips. The parameters θ^c and $\theta^{\bar{c}}$ are exogenous to the design problem. So are the quantities O_i^c, $O_i^{\bar{c}}$, D_j^c, $D_j^{\bar{c}}$. The matrix $B = [b_{ik}]$ represents the site costs as in section II.

B. Solution algorithms

Two heuristic techniques of the hill-climbing type were designed and implemented to solve the land-use design problem (17) - (29). They can be considered as generalizations of ALLCØS and ALLCØU respectively.

1. First heuristic technique: ALCDA2

This hill-climbing technique produces a local optimum and works as follows:

a) Initialization: generate a random allocation of the n activities. By solving CDA and DCG we obtain the equilibrium values for g_{pq}^c, $g_{pq}^{\bar{c}}$, u_{pq} and therefore we can compute the current value of the objective function.

b) We assume that g_{pq}^c, $g_{pq}^{\bar{c}}$ (all p,q) and u_{pq} (all p,q) are fixed at their current values. We then obtain a quadratic assignment problem with two types of interaction. For each one of the n(n-1)/2 possible pairwise interchanges of activities we can compute an estimate of the change in the cost of the spatial allocation, were the interchange to be implemented. (It is only an *estimate* because of the assumption that the pairwise demands and costs remain fixed during the interchange.)

c) We order the possible interchanges in the order of non-decreasing values of the estimates. For the first ME (where ME is a parameter of the

program) interchanges in the list we compute the true value of the objective function after interchange by solving CDA and DCG, and if the best interchange leads to an improvement of the objective function over the value of the current solution we implement that interchange and return to step b. If none of the interchanges in the list leads to an improvement over the current solution, the algorithm stops.

The doubly constrained gravity model was solved by a balancing algorithm and the combined distribution assignment by an algorithm due to Evans (1976).

2. Second heuristic technique: ALCDA

This heuristic is also a hill-climbing technique which produces a local optimum. It works by iterating between a pairwise interchange procedure for the quadratic assignment problem and a simultaneous application of the doubly-constrained gravity model and of the combined distribution assignment. It proceeds as follows:

a) Initialization: generate a random spatial allocation of the n activities and compute the equilibrium demands g_{pq}^{c}, $g_{pq}^{\bar{c}}$ and costs u_{pq}, associated with this initial solution. Compute the value of the objective function.

b) Assuming that the peak-hour travel costs between origin-destination pairs, the peak-hour and non peak-hour demands between O-D pairs remain constant and equal to their current values we compute a local optimum to the resulting quadratic assignment problem by applying a pairwise interchange technique.

c) Compute the true equilibrium peak-hour travel costs, peak-hour and non peak-hour demands by applying CDA and DCG to the current permutation of activities. Compute the associated true value of the objective function. If there is an improvement, update the current best spatial

allocation and go to step b. Otherwise, i.e. if no improvement occurs, the algorithm stops.

In step b, a code called QSWAP, developed by Britton Harris and generalizing ACTIV for the solution of quadratic assignment problems with multiple types of interactions, was implemented; the doubly-constrained gravity model and the combined distribution assignment problem were solved as in ALCDA2.

C. Numerical experiments

The two algorithms ALCDA2 and ALCDA were implemented on the CDC Cyber 173 of the Université de Montréal. Table 4 reports the results with ALCDA2. It was found that the quality of the solutions does not improve with the size of the parameter ME and that the method is inefficient as evidenced by the computer times. On the other hand ALCDA is more efficient. By comparing the local optima generated by ALCDA and ALCDA2 (see tables 4 and 5), it appears that ALCDA produces solutions of better (or equal) quality than ALCDA2 five times out of six. Therefore ALCDA is clearly the better method of the two. When it is applied without congestion (see table 6) it is even more efficient since the algorithm reduces to iterations between a gravity model and an algorithm for the quadratic assignment problem.

Value of the initial solution	Value of the local optimum	Execution time in seconds of CDC Cyber 173	Number of exchanges implemented
345 332	138 346	47.2	5
293 327	129 527	31.4	3
267 036	124 907	32.0	3
243 584	125 310	32.0	3
424 286	133 030	42.3	4
424 420	128 392	41.2	4

Table 4

Results obtained with ALCDA2 on the data set # 3 with six different initial permutations generated at random with $ME = 1$, $\theta^c = \bar{\theta}^c = 1.5$ [1,2]

1. In Evans' method 20 iterations were the maximum allowed and the convergence parameter ε was .001 in the gravity model.

2. In the experiments reported here, it was assumed that trips are evenly divided between peak-hours and non-peak-hours.

Value of the initial solution	Value of the local optimum	Execution time in seconds of CDC Cyber 173	Number of exchanges implemented
345 332	106 352	23.7	11
293 327	129 527	15.9	3
267 036	131 205	15.5	4
243 584	125 310	15.7	3
424 286	127 761	25.4	5
424 420	122 434	15.9	8

Table 5

Results obtained with ALCDA on the data set # 3 with six different initial permutations generated at random, $\theta^c = \theta^{\bar{c}} = 1.5$ [1,2]

1. In Evans' method, 20 iterations were the maximum allowed and the convergence parameter ε was .001 in the gravity model.

2. Trips were evenly divided between peak-hours and non-peak-hours.

Value of the initial solution	Value of the local optimum	Execution time in seconds of CDC Cyber 173	Number of iterations
325 896	111 148	1.0	9
282 580	102 384	0.9	5
251 439	116 353	0.8	4
233 503	114 324	0.8	3
414 228	116 272	1.2	5
412 418	113 160	1.0	8

Table 6

Results obtained with ALCDA on the data set # 3 with six different initial permutations generated at random,

$\theta^C = \theta^{\bar{C}} = 1.5$

(it is assumed that all trips are non-peak-hour trips)

IV Discussion and conclusion

This paper presented land-use design models formulated as optimization problems and including a more realistic representation of transportation behavior (automobile congestion and variable demand for travel). Numerically promising algorithms were developed and tested to solve the resulting optimization problems.

Several extensions of the approach are possible by relaxing some of the simplifying assumptions. For instance the assumption of one mode of transport could be relaxed if we replaced in ALCDA the combined distribution assignment model by either a general bimodal equilibrium model such as EMME (Florian *et al.*, 1977 and 1978) or by the combined modal split trip distribution, trip assignment model (Florian and Nguyen, 1978). The assumption that the network is given can also be relaxed: for instance we can replace QSWAP in ALCDA by an algorithm for the joint optimization of land use and transportation such as the algorithm IMPRØ1 in Los (1978). Finally, we can make the objective function of the land use design model with variable demand more realistic by replacing the minimization of travel cost by the maximization of an accessibility indicator (Coelho and Williams, 1977), and by introducing multiple types of interaction costs, such as costs due to air pollution. The latter extension requires simply that QSWAP in ALCDA be used with more than two interactions types (Hopkins and Los, 1979).

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APPENDIX 1

The data

A. The network:

Figure 1 presents the network used throughout the numerical experiments. It has 56 links and 25 nodes. However only 9 nodes among them, situated at the centroids of nine square zones can be assigned activities, i.e. can generate or attract traffic. Each link a is characterized by a volume-delay function of the following form:

$$t_a = t_o \left[1.0 + 0.15 \left(\frac{v}{c}\right)^4 \right]$$

where t_a is the travel time on link a for a flow v (in hours), t_o is the free-flow travel time and c is the practical capacity of link a (in number of trips). The parameters t_o and c vary with each link a. Table A1 gives the link characteristics.

Different data sets were used for the demand for travel.

Data set # 1

The trips between activities are given by Table A2. The site costs are zero for all locations.

Data set # 2

The site costs are zero and the trips between activities are given by Table A3.

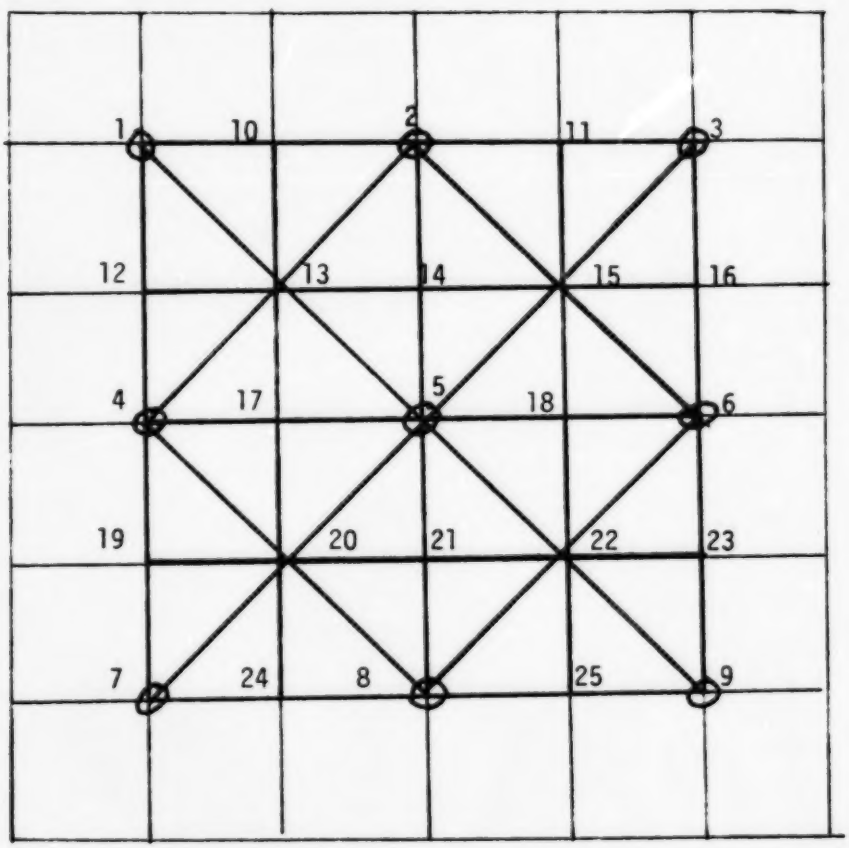

Figure 1

The network and locations of the numerical example

		t_0	c
1	10	4.62	967
1	12	3.78	854
1	13	2.16	784
2	10	2.23	556
2	11	3.57	754
2	13	4.29	870
2	14	4.32	559
2	15	3.88	463
3	11	3.05	330
3	15	3.18	618
3	16	4.17	652
4	12	.64	923
4	13	2.90	919
4	17	4.39	324
4	19	1.23	686
4	20	3.10	804
5	13	4.93	738
5	14	1.50	490
5	15	1.26	826
5	17	3.85	701
5	18	1.78	957
5	20	2.65	701
5	21	1.93	726
5	22	4.24	619
6	15	3.72	919
6	16	3.77	662
6	18	4.12	903
6	22	4.95	518
6	23	2.53	907
7	19	4.70	369
7	20	.66	827
7	24	1.58	655
8	20	2.07	331
8	21	4.56	433
8	22	3.88	734
8	24	2.18	738
8	25	.70	649
9	22	.67	803
9	23	2.73	956
9	25	4.34	482
10	1	4.62	967
10	2	2.23	556
10	13	3.69	525
11	2	3.57	754
11	3	3.05	330

		t_0	c
11	15	.61	338
12	1	3.78	854
12	4	.64	923
12	13	4.04	424
13	1	2.16	784
13	2	4.29	870
13	4	2.90	919
13	5	4.93	738
13	10	3.69	525
13	12	4.04	424
13	14	1.91	854
13	17	3.05	785
14	2	4.32	559
14	5	1.50	490
14	13	1.91	854
14	15	2.14	625
15	2	3.88	463
15	3	3.18	618
15	5	1.26	826
15	6	3.72	919
15	11	.61	338
15	14	2.14	625
15	16	1.54	644
15	18	4.63	661
16	3	4.17	652
16	6	3.77	662
16	15	1.54	644
17	4	4.39	324
17	5	3.85	701
17	13	3.05	785
17	20	.50	797
18	5	1.78	957
18	6	4.12	903
18	15	4.63	661
18	22	4.03	904
19	4	1.23	686
19	7	4.70	369
19	20	3.35	591
20	4	3.10	804
20	5	2.65	701
20	7	.66	827
20	8	2.07	331
20	17	.50	797

		t_0	c
20	19	3.35	591
20	21	3.83	377
20	24	2.01	966
21	5	1.93	726
21	8	4.56	433
21	20	3.83	377
21	22	2.12	439
22	5	4.24	619
22	6	4.95	518
22	8	3.88	734
22	9	.67	803
22	18	4.03	904
22	21	2.12	439
22	23	1.68	799
22	25	4.90	757
23	6	2.53	907
23	9	2.73	956
23	22	1.68	799
24	7	1.58	655
24	8	2.18	738
24	20	2.01	966
25	8	.70	649
25	9	4.34	482
25	22	4.90	757

Table A1

Link characteristics of the numerical example

0	222	55	44	22	33	666	444	444
6	0	444	444	555	4	555	555	333
3	26	0	77	44	22	616	1000	8
1009	630	377	0	633	744	383	876	50
66	77	63	50	0	97	87	108	844
622	966	62	855	971	0	976	87	86
7	6	62	75	75	97	0	9	8
385	850	85	97	101	99	75	0	622
7	5	77	66	44	55	666	623	0

Table A2

Trips between activities
for data set # 1

0	114	29	526	44	327	336	414	225
114	0	235	537	316	485	280	702	169
29	235	0	227	53	42	339	542	42
526	537	227	0	341	799	229	486	58
44	316	53	341	0	534	81	104	444
327	485	42	799	534	0	536	93	70
336	280	339	229	81	536	0	42	337
414	702	542	486	104	93	42	0	622
225	169	42	58	444	70	337	622	0

Table A3

Trips between activities
for data set #2

Data set # 3

Total number of trips generated by each origin:

$$O_p = (1930, 2896, 1796, 4702, 1392, 4625, 339, 2314, 1543)$$

Total number of trips attracted by each destination:

$$D_q = (2105, 2782, 1225, 1708, 2445, 1151, 4024, 3702, 2395)$$

The proportion of the total number of trips generated (or attracted) by each activity occurring during peak-hours is given *exogenously* for each problem. The site costs are given by Table A4.

0	22200	5500	4400	2200	3300	66600	44400	44400
600	0	44400	44400	55500	400	55500	55500	33300
300	2600	0	7700	4400	2200	61600	100000	800
100900	63000	37700	0	63300	74400	38300	87600	5000
6600	7700	6300	5000	0	9700	8700	10800	84400
62200	96600	6200	85500	97100	0	97600	8700	8600
700	600	6200	7500	7500	9700	0	900	800
38500	85000	8500	9700	10100	9900	7500	0	62200
700	500	7700	6600	4400	5500	66600	62300	0

Table A4

Site costs
for data set # 3

APPENDIX 2

Proof that the estimate used in the heuristic technique ALLCØS is a lower bound on the cost after exchange of activities

The cost change due to site costs is computed exactly in the estimate. Therefore we simply have to prove that the estimate of the change in total transportation cost after exchange of two activities is a lower bound for the true change in that cost. We use slightly different notation than in the main text. Let ρ be the mapping of the set of activities into the set of locations corresponding to a given spatial allocation. The mapping ρ is expressed by a permutation of the first n integers: $\rho(i)$, $i = 1...n$ is the location assigned to activity i in the mapping ρ.

$$\phi(\rho) = \sum_{a \in A} v_a^* \; s_a(v_a^*)$$

where the v_a^* are the system-optimal flows on the links for a given ρ[1]

Some mathematical results from duality theory have to be proved first. Consider the two minimization problems.

$$\underline{P1} \qquad \min_{x \geq 0} f(x) \qquad \text{s. t.} \quad Ax = 0$$

$$\underline{P2} \qquad \min_{x \geq 0} f(x) \qquad \text{s. t.} \quad Ax = y$$

where $f(x)$ is a convex differentiable function of all variables in the vector x, A is a matrix and y a perturbation vector of the R.H.S. of the linear constraints. Notice P1 and P2 differ only in the right hand side of the linear constraints.

---

1 ρ does not appear in the objective function, but in the constraints.

Consider the Lagrangian function of P1, and let (x^*, u^*) be the saddle point for P1.

We have:

$$\mathcal{L}(x^*, u^*) = f(x^*) + u^{*t}Ax^*$$

and $$\mathcal{L}(x^*, u^*) \leq \mathcal{L}(x, u^*) \quad \forall x \geq 0$$

or $$f(x^*) \leq f(x) + u^{*t}Ax \quad \forall x \geq 0 \qquad (E)$$

Let x^1 be the optimal solution of P2.

We have $Ax^1 = y$

(E) implies: $f(x^*) \leq f(x^1) + u^{*t} Ax^1$

or $f(x^*) \leq f(x^1) + u^{*t}y$

$\Rightarrow$ $- u^{*t}y \leq \Delta f$ (See 1)

where Δf is the change in the optimal value of the minimization problem after perturbation of the R.H.S. of the linear equalities, and u^{*t} is the row vector of dual variables of the saddle point of P1.

We can apply directly these results with ϕ instead of f and

$$q_{\rho^{-1}(k)\rho^{-1}(\ell)} \qquad -\sum_m h_{m,k\ell} = 0$$

as linear constraints.

$(-y)$ is represented here by the column vector of the changes $\Delta q_{k\ell}$ in the number of trips between location pairs.

u^{*t} is the row vector of the network equilibrium dual variables before exchange.

---

1 Strickly speaking, this inequality is true for any feasible solution x^1 to P2, but Δf is minimum for the optimal solution to P2.

The scalar product $-u^{*t}y$ is represented here by $\sum_k \sum_\ell u^*_{k\ell} \Delta q_{k\ell}$

Δf is the change in the objective function after exchange, i.e. $\Delta \phi$.

Hence the inequality.

$$\sum_K \sum_\ell u^*_{k\ell} \Delta q_{k\ell} \leq \Delta \phi$$

q. e. d.

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